1. Scalar field

Having

$$S = -\int d^4x\sqrt{-g}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + U(\phi)\right) = -\int d^4x\mathcal{L}$$

where

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + U(\phi)\right)$$

(a) Let's use Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi)} = \sqrt{-g}g^{\sigma\nu}\partial_\nu\phi, \quad \partial_\sigma\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi)} = \partial_\sigma\left(\sqrt{-g}g^{\sigma\nu}\partial_\nu\phi\right) \tag{1}$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = \sqrt{-g}U'(\phi) \tag{2}$$

This gives

$$\frac{1}{\sqrt{-g}}\partial_\sigma\left(\sqrt{-g}g^{\sigma\nu}\partial_\nu\phi\right) = U'(\phi) \tag{3}$$

This can be conveniently written as

$$\nabla_\sigma\nabla^\sigma\phi = U'(\phi)$$

or

$$\nabla^2\phi = U'(\phi)$$

For stress-energy tensor the formula

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}$$

becomes handy. Using product rule one has

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{2}{\sqrt{-g}}\left(\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}\left(\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + U(\phi)\right) + \frac{1}{2}\sqrt{-g}\partial_\mu\phi\partial_\nu\phi\right)$$

With

$$\frac{\partial g}{\partial g^{\mu\nu}} = -g_{\mu\nu}g, \quad \frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu} \tag{4}$$

one gets

$$T_{\mu\nu} = \partial_\mu\phi\;\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\left(\partial\phi\right)^2 + U(\phi)\right) \tag{5}$$

(b) Assuming homogenity, the energy-momentum tensor is

$$T_{\mu\nu} = \partial_\mu\phi\;\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{tt}\left(\partial_t\phi\right)^2 + U(\phi)\right) \tag{6}$$

or in upper components

$$T^{\mu\nu} = \partial^\mu\phi\;\partial^\nu\phi - g^{\mu\nu}\left(\frac{g^{tt}}{2}\left(\partial_t\phi\right)^2 + U(\phi)\right) \tag{7}$$

For a perfect fluid, $T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu}$, and $U = \sqrt{-g^{tt}}\partial_t$ because the field is at rest. Then, one can read off

$$p = -\frac{1}{2}g^{tt}\left(\partial_t\phi\right)^2 - U(\phi) \tag{8}$$

$$\rho = -\frac{1}{2}g^{tt}(\partial_t\phi)^2 + U(\phi) \tag{9}$$

In slow-roll approximation $(\partial_t\phi)^2 \ll U(\phi)$ that leads to $p = -\rho$, which is the equation of state of dark energy.

2. Lemaître Coordinates

(a) As usual, we define the conserved quantity

$$E = -\frac{dt}{d\tau}g_{tt} = \frac{dt}{d\tau}\left(1 - \frac{r_S}{r}\right). \tag{10}$$

For a radially infalling observer we have

$$-1 = -E^2\left(1 - \frac{r_S}{r}\right)^{-1} + \left(1 - \frac{r_S}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2. \tag{11}$$

Reshuffling terms,

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - 1 + \frac{r_S}{r}. \tag{12}$$

The initial condition that $\frac{dr}{d\tau}(0) = 0$ fixes $E = 1$ since the observer is infinitely far away. We thus have that the components of the four velocity of the observer are

$$U^t = \frac{dt}{d\tau} = \left(1 - \frac{r_S}{r}\right)^{-1}, \qquad U^r = \frac{dr}{d\tau} = -\sqrt{\frac{r_S}{r}}. \tag{13}$$

(b) Given that all components depend only on the radial coordinate r, we have

$$[U, V] = U^r(\partial_r V^t)\partial_t + U^r(\partial_r V^r)\partial_r - V^r(\partial_r U^r)\partial_r - V^r(\partial_r U^t)\partial_t. \tag{14}$$

For this to vanish, we thus need

$$U^r\partial_r V^t = V^r\partial_r U^t, \qquad U^r\partial_r V^r = V^r\partial_r U^r. \tag{15}$$

Substituting the functions we have for U^r and U^t we obtain

$$\begin{aligned} -\sqrt{\frac{r_S}{r}}\partial_r V^r &= \frac{1}{2}V^r\sqrt{\frac{r_S}{r^3}} \\ \partial_r V^r &= -\frac{1}{2r}V^r \end{aligned} \tag{16}$$

which has solution $V^r = C\sqrt{\frac{r_s}{r}}$, Continuing with perpendicularity condition

$$0 = V_\mu U^\mu = g_{tt} V^t U^t + g_{rr} V^r U^r$$
$$0 = -V^t - \frac{\sqrt{\frac{r_s}{r}} V^r}{1 - \frac{r_s}{r}} = -V^t - C \frac{\frac{r_s}{r}}{1 - \frac{r_s}{r}}$$

giving $V^t = -\frac{C}{1-\frac{r_s}{r}} \frac{r_s}{r}$

Quick checkup confirms $U^r \partial_r V^r = V^r \partial_r U^r$ is obeyed as well. As there is no reason to complicate things, C will be set to 1.

(c) Let's start by computing the dot products. As imposed before, $V_\mu U^\mu = 0$, therefore $g_{\tau\rho} = 0$. Other combinations are

$$U_\mu U^\mu = -\frac{1}{1 - \frac{r_s}{r}} + \frac{1}{1 - \frac{r_s}{r}} \frac{r_s}{r} = -1 \tag{17}$$
$$V_\mu V^\mu = -\frac{1}{1 - \frac{r_s}{r}} \left(\frac{r_s}{r}\right)^2 + \frac{1}{1 - \frac{r_s}{r}} \frac{r_s}{r} = \frac{r_s}{r} \tag{18}$$

Dual one-forms are found by lowering indices:

$$\mathrm{d}\tau = \frac{1}{U^2} U_\mu dx^\mu = \mathrm{d}t + \left(1 - \frac{r_s}{r}\right)^{-1} \sqrt{\frac{r_s}{r}} \mathrm{d}r \tag{19}$$
$$\mathrm{d}\rho = \frac{1}{V^2} V_\mu dx^\mu = \mathrm{d}t + \left(1 - \frac{r_s}{r}\right)^{-1} \sqrt{\frac{r}{r_s}} \mathrm{d}r \tag{20}$$

One can invert such a relation for $\mathrm{d}r, \mathrm{d}t$ in terms of new coordinates and plug it in the metric, however, much quicker way is just reading off the norm of basis vectors. Therefore,

$$ds^2 = -\mathrm{d}\tau^2 + \frac{r_s}{r} \mathrm{d}\rho^2 \tag{21}$$

Note: It feels weird to 'read off' the coefficients, but remember, $d\rho$ is something that gives 1 when acting on ∂_ρ and so on. And as $V_\mu V^\mu$ is basis independent, the metric needs suitable prefactors. But do check the explicit algebra, dear student of mine.

(d) As (ρ, τ) forms coordinate system, r can be simply get from integrating (12). The computation gives

$$\frac{dr}{d\tau} = -\sqrt{\frac{r_s}{r}} \tag{22}$$
$$\frac{2 r^{\frac{3}{2}}}{3 r_s^{\frac{1}{2}}} = -\tau + c(\rho) \tag{23}$$

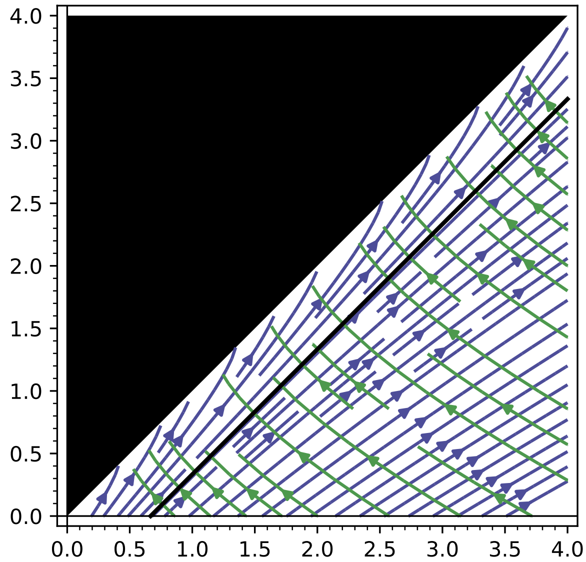
Imposing boundary conditions give $c(\rho) = \rho$, which gives

$$r = \left(\frac{3}{2} (\rho - \tau)\right)^{\frac{2}{3}} r_s^{\frac{1}{3}} \tag{24}$$

(e) The points of interests are singularity $r = 0$ at $\rho = \tau$ and event horizon $r = r_s$, giving $\rho - \tau = \frac{2}{3} r_s$. This will be confirmed by finding the light cones via solving $ds^2 = 0$:

$$\frac{d\rho}{d\tau} = \pm\sqrt{\frac{r}{r_s}} = \pm\sqrt{\frac{r}{r_s}} = \pm\sqrt[3]{\frac{3(\rho - \tau)}{2 r_s}} \tag{25}$$

One may try solving this bloodbath of differential equation, however the most important feature is prominent – right-going light cone is tangent to $r = r_s$ line, which confirms r_s is an event horizon. This is depicted on following $\rho - \tau$ diagram, where black region marks singularity, and black line denotes event horizon.



(f) This line corresponds to a body freefalling into a black hole – we obtained it by (indirectly) solving the geodesics equation in $r - t$ coordinates. One can explicitly compute Christoffel $\Gamma^{\rho}_{\tau\tau}$ to confirm that. The proper time is obtained by integrating the line element over line from $\tau = -\frac{2}{3}r_s$ (horizon) to $\tau = 0$ (singularity), giving proper time of $\frac{2}{3}r_s$.

(g) The minimal proper time is given by following trajectory closest to left-going light-like trajectory. As this is line of finite (euclidian) length, but the tangent vectors are close to null-like, the lower bound for this time is 0.

The maximal proper time is slightly more involved. Let's write the infall time as the solution to (12):

$$\tau = \int_0^{r_s} \frac{\mathrm{d}r}{\sqrt{E^2 - 1 + \frac{rs}{r}}} \tag{26}$$

The integrand is strictly decreasing as a function of E, therefore, the proper time to fall into the singularity is maximized at smallest E^2, which is 0. Therefore, the (easier) integral

$$\tau = \int_0^{r_s} \frac{\mathrm{d}r}{\sqrt{-1 + \frac{rs}{r}}} = \frac{\pi}{2} r_s \tag{27}$$

3. Extra dimension

(a) Making a coordinate transformation $\phi' = \phi + f(x)$ results in

$$\mathrm{d}\phi' = \mathrm{d}\phi + \partial_\mu f(x) \mathrm{d}x^\mu$$

Plugging that into the metric gives

$$ds^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + R^2(\mathrm{d}\phi' - \partial_\mu f(x)\mathrm{d}x^\mu + A_\mu \mathrm{d}x^\mu) \tag{28}$$

or

$$ds^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + R^2(\mathrm{d}\phi' + A'_\mu \mathrm{d}x^\mu) \tag{29}$$

with $A'_\mu = A_\mu - \partial_\mu f(x)$.

This, (for a good reason), looks like gauge transform of EM field.

(b) Einstein-Hilbert action is $S = \int \sqrt{-g} R dx^4$, and Ricci scalar is schematically (skipping indices)

$$R \sim \Gamma \cdot \Gamma + \partial\Gamma \sim (\partial g)(\partial g) + \partial(\partial g)$$

However, in this metric the EH action depends only on A_μ, and, as the $A_\mu \to A_\mu + \partial_\mu f(x)$ is a coordinate transformation, Ricci scalar may only depend on local, gauge invariant functions of A.

From electromagnetism, one knows that such objects are constructed from tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Therefore, the dependence of $\sqrt{-g}$ on A is trivial (there are no local gauge-invariant terms containing A without derivatives), and structure of R implies EH action must be quadratic in F.

Therefore one can limit possible terms to

$$S = \int \alpha\, F_{\mu\nu}F^{\mu\nu} + \beta\, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma} \mathrm{d}x^4 \tag{30}$$

However, under coordinate transformation $\phi \to -\phi$

$$\begin{aligned} A_\mu &\to -A_\mu \\ F_{\mu\nu} &\to -F_{\mu\nu} \\ \varepsilon^{\mu\nu\rho\sigma} &\to -\varepsilon^{\mu\nu\rho\sigma} \end{aligned}$$

which leaves the α-term invariant ('true' scalar), but changes the sign of the β term (pseudoscalar). Therefore Einstein-Hilbert action must be

$$S = \int \alpha\, F_{\mu\nu}F^{\mu\nu} \mathrm{d}x^4 \tag{31}$$

which is precisely one of electromagnetism.

Careful reader may also notice that geodesics equation provides Lorentz force with 'charge' being proportional to momentum in ϕ direction. Such unification of electromagnetism and gravity is called Kaluza-Klein theory, and may be used to describe other gauge theories in terms of diffeomorphisms.

For ones wanting to learn more, there is an amazing paper describing possibility of using Kaluza-Klein mechanism for unifying Standard Model with GR.